

SOME ASPECTS OF GENERALIZED COVERING SPACE THEORY

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ABSTRACT

THESIS: Some Aspects of Generalized Covering Space Theory

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Covering space theory is a classical tool used to characterize the geometry and topology of real or abstract spaces. It seeks to separate the main geometric features from certain algebraic properties. For each conjugacy class of a subgroup of the fundamental group, it supplies a corresponding covering of the underlying space and encodes the interplay between algebra and geometry via group actions.

The full applicability of this theory is limited to spaces that are, in some sense, locally simple. However, many modern areas of mathematics, such as fractal geometry, deal with spaces of high local complexity. This has stimulated much recent research into generalizing covering space theory by weakening the covering requirement while maintaining most of the classical utility.

This project focuses on the relationships between generalized covering projections, fibrations with unique path lifting, separation properties of the fibers, and continuity of the monodromy.

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Chapter 1

Results from Classical Theory

We assume familiarity with the basic notions of point set topology and the fundamental group, as presented in [12]. The classical way of defining a covering projection is through the use of a local homeomorphism condition. We begin by providing the definition of a classical covering projection, as defined in [14], and some well known examples of covering spaces. Then we introduce the notion of unique path lifting and restate the lifting criterion for covering spaces found in [14].

Definition 1.1 (Evenly Covered). Let $p : E \rightarrow X$ be a continuous surjective map between topological spaces. An open set U of X is said to be *evenly covered* by p if the inverse image $p^{-1}(U)$ can be written as the union $\cup_{i \in I} V_i$ of disjoint open sets V_i in E such that for each i , the restriction of p to V_i is a homeomorphism of V_i onto U .

Definition 1.2 (Covering Space and Covering Projection). Let $p : E \rightarrow X$ be a continuous and surjective map between topological spaces. If every point $x \in X$ has a neighborhood U that is evenly covered by p , then p is called a *covering projection*, and E is said to be a *covering space* of X .

The real line \mathbb{R} is a covering space for the circle S^1 with covering projection $p(t) = (\cos t, \sin t)$. The plane $\mathbb{R} \times \mathbb{R}$, and the infinite cylinder $\mathbb{R} \times S^1$, are both covering spaces for the torus $S^1 \times S^1$.

Definition 1.3 (Unique Path Lifting (UPL)). A continuous surjection $p : E \rightarrow X$ is said to have *unique path lifting* if, for every $f, g : [0, 1] \rightarrow E$ such that $p \circ f = p \circ g$ and $f(0) = g(0)$, we have $f = g$.

Lemma 1.4 (Lifting Criterion). *Let $p : E \rightarrow X$ be a covering projection with $p(e_0) = x_0$. Suppose $f : Y \rightarrow X$ is a continuous map such that $f(y_0) = x_0$ and such that Y is connected and locally path-connected. Then:*

There is a “lift” $f' : Y \rightarrow E$ such that $p \circ f' = f$ and $f'(y_0) = e_0 \iff f_{\#}(\pi_1(Y, y_0)) \leq p_{\#}(\pi_1(E, e_0))$.

In addition, if such a lifting exists, then it is unique.

As discussed in [14], the lifting criterion shows that all paths and their homotopies lift uniquely in a covering projection. In particular, $p_{\#} : \pi_1(E, e_0) \rightarrow \pi_1(X, x_0)$ is injective. In addition, [14] also provides a classification for when covering projections exist that preserve given subgroups of the fundamental group. The proof Spanier gives in [14, Theorem 2.5.13] is especially of note, since this is related to the construction in Definition 2.2.

The following lemma combines [14, Lemma 2.5.11] and [14, Theorem 2.5.13].

Lemma 1.5. *Assume X is connected and locally path-connected, $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. There exists a covering projection $p : (E, e_0) \rightarrow (X, x_0)$ with $p_{\#}(\pi_1(E, e_0)) = H$ if and only if there exists an open cover \mathcal{U} of X such that $\pi(\mathcal{U}, x_0) \leq H$, where $\pi(\mathcal{U}, x_0)$ is the (normal) subgroup of $\pi_1(X, x_0)$ generated by homotopy classes of closed paths with a representative of the form $(\alpha * \beta) * \alpha^-$, where β is a closed path in some element of \mathcal{U} and α is a path from x_0 to $\beta(0)$. (Here $\alpha^-(t) = \alpha(1-t)$ denotes the reverse path of α .)*

This lemma shows that the Hawaiian Earring, formed in the Euclidean plane as the union of all circles C_n with radius $\frac{1}{n}$ and center $(0, \frac{1}{n})$, does not have a covering projection $p : (E, e_0) \rightarrow (X, x_0 = (0, 0))$ with $p_{\#}(\pi_1(E, e_0)) = 1$, because for any open cover \mathcal{U} of X , there is an element of the cover $U \in \mathcal{U}$ with $x_0 \in U$, and $C_n \subseteq U$ for some n . Since a loop around this circle C_n is not nullhomotopic in the Hawaiian Earring, U cannot be evenly covered with a simply connected covering space E .

In addition to generalizing the concept of a covering projection, we are also interested in extending our results to consider fibrations: maps which have the homotopy lifting property with respect to every space.

Definition 1.6 (Homotopy Lifting Property). A continuous map $p : E \rightarrow X$ is said to have the *homotopy lifting property with respect to a space Y* if, given continuous maps $f' : Y \rightarrow E$ and

$F : Y \times [0, 1] \rightarrow X$ such that $F(y, 0) = p \circ f'(y)$ for all $y \in Y$, there is a map $F' : Y \times [0, 1] \rightarrow E$ such that $F'(y, 0) = f'(y)$ for all $y \in Y$ and $p \circ F' = F$.

The homotopy lifting property with respect to $Y = \{y_0\}$ is also called the path lifting property.

Definition 1.7 (Fibration). A continuous map $p : E \rightarrow X$ is called a *fibration* if p has the homotopy lifting property with respect to every space. We call E the *total space*, and X is called the *base space* of the fibration. For $x \in X$, $p^{-1}(x) = p^{-1}(\{x\})$ is called the *fiber* over x .

We now define monodromy as it relates to covering projections, which allows us to map one fiber to another in the covering space in such a way that the map between the fibers respects a given path between the images of the two fibers.

Definition 1.8 (Monodromy). Let $p : E \rightarrow X$ be a fibration with unique path lifting, $y, z \in X$, and $\beta : [0, 1] \rightarrow X$ a path that starts at y and ends at z . We now define the *monodromy* $\phi_\beta : p^{-1}(y) \rightarrow p^{-1}(z)$ as follows: For $y' \in p^{-1}(y)$, let $\beta' : [0, 1] \rightarrow E$ be the unique lift with $p \circ \beta' = \beta$ and $\beta'(0) = y'$. We define $\phi_\beta(y') = \beta'(1)$.

Remark 1.9. Spanier shows that every covering projection is a fibration with unique path lifting [14, Lemma 2.2.3] and shows that a fibration has unique path lifting if and only if it has totally path-disconnected fibers [14, Theorem 2.2.5]. Spanier also shows that in a fibration with unique path lifting, all monodromies are continuous [14, Theorem 2.3.7] and hence, all fibers are homeomorphic [14, Corollary 2.3.8].

Recall the following fact from [14, Corollary 2.2.13]: If $p_1 : E_1 \rightarrow X$ and $p_2 : E_2 \rightarrow X$ are two covering projections of a locally connected space X and $f : E_1 \rightarrow E_2$ a surjective map with $p_2 \circ f = p_1$, then $f : E_1 \rightarrow E_2$ is a covering projection.

Definition 1.10. We say that two covering projections $p_1 : E_1 \rightarrow X$ and $p_2 : E_1 \rightarrow X$ are *equivalent* if there is a homeomorphism $f : E_1 \rightarrow E_2$ such that $p_2 \circ f = p_1$. When $E_1 = E_2 = E$ and $p_1 = p_2 = p$, we call such a homeomorphism f an *automorphism* of $p : E \rightarrow X$. (Note that the set $\text{Aut}(E \xrightarrow{p} X)$ of automorphisms of $p : E \rightarrow X$ forms a group under function composition.)

Next, we recall the definition of a universal covering space, and state some corollaries that, when taken together, classifies every covering projection onto a base space by the projected fundamental group, and in particular, we see that if a space has a universal covering space, then any other universal covering space is equivalent.

Definition 1.11 (Universal Covering Space). Let E be connected and $p : E \rightarrow X$ be a covering projection. We call E a *universal covering space*, if for any covering projection $r : Y \rightarrow X$ with Y connected, there exists a covering projection $q : E \rightarrow Y$ such that $p = r \circ q$.

We quote the following from [14, Corollary 2.5.3]:

Corollary 1.12. *Two connected covering spaces of a connected, locally path-connected space are equivalent if and only if their fundamental groups, generated over the same base point, map to conjugate subgroups.*

We refer to $p : (E, e_0) \rightarrow (X, x_0)$ as the covering projection with respect to $H = p_{\#}(\pi_1(E, e_0)) \leq \pi_1(X, x_0)$ if it exists.

From the lifting criterion, one obtains the following two well-known corollaries:

Corollary 1.13. *If X is connected and locally path-connected, $p : E \rightarrow X$ is a covering projection, and E is simply connected, then E is a universal covering space. For locally path-connected X , X has a simply connected covering space if and only if X is semilocally simply connected, i.e., if and only if $\pi_1(\mathcal{U}, x_0) = 1$ for some open cover \mathcal{U} of X .*

Corollary 1.14. *If X is connected and locally path-connected, X has a universal covering space if and only if the collection $\{\pi_1(\mathcal{U}, x_0) : \mathcal{U} \text{ an open cover of } X\}$ has a minimal element.*

For example, the Hawaiian Earring does not have a universal covering space, since for every open cover \mathcal{U} , there is a refinement \mathcal{V} of \mathcal{U} with $\pi_1(\mathcal{V}, x_0) \subsetneq \pi_1(\mathcal{U}, x_0)$.

In the next section, we define a generalization of covering projection which aims to keep the classical utility of covering projections defined above while removing the local homeomorphism limitation.

Chapter 2

Generalized Covering Space Theory

We now introduce a generalization of a covering projection, which is not defined by evenly covered neighborhoods, that was first defined in [9] and later generalized further in [2]. Comparing this definition to Lemma 1.4 above shows that this new definition is a generalization of a covering space.

Definition 2.1 (Generalized Covering). Let X be a path-connected topological space. A surjective map $p : \tilde{X} \rightarrow X$ is a *generalized covering projection* if

1. \tilde{X} is connected and locally path-connected,
2. for every $\tilde{x} \in \tilde{X}$, every connected and locally path-connected space Y , and every map $f : (Y, y) \rightarrow (X, p(\tilde{x}))$ such that $f_{\#}(\pi_1(Y, y)) \leq p_{\#}(\pi_1(\tilde{X}, \tilde{x}))$, there is a unique lift $\tilde{f} : (Y, y) \rightarrow (\tilde{X}, \tilde{x})$ such that $p \circ \tilde{f} = f$.

There are many topological spaces which have simply connected, and therefore universal, generalized covering spaces. For example, if $X \subseteq \mathbb{R} \times \mathbb{R}$, or if X is topologically one-dimensional, then X has a simply connected generalized covering space. More generally, X has a simply connected generalized covering space if the natural homomorphism $\pi_1(X, x) \rightarrow \tilde{\pi}_1(X, x)$ to the first Čech homotopy group is injective [9], or if X is a metric space with $\pi_1(X, x)$ residually free [4]. In particular, the Hawaiian Earring, the Menger Sponge and the Sierpinski carpet all have simply connected generalized covering spaces. For the Hawaiian Earring, this generalized covering projection cannot be a fibration by Remark 1.9; this projection does not have homeomorphic fibers.

Constructing a generalized covering space for a given base space follows the same procedure Spanier follows when constructing covering spaces as in the proof of Lemma 1.5. We define the collection of homotopic paths relative to H and the topology on this space as follows:

Definition 2.2 (Path Space and Endpoint Projection). Let X be a path-connected space, $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. We say that two paths α and β in X that both start at x_0 are *equivalent with respect to H* if $\alpha(1) = \beta(1)$ and $[\alpha * \beta^-] \in H$.

Define \tilde{X}_H to be the set of all such equivalence classes $\langle \alpha \rangle$, and define a basis for \tilde{X}_H as the collection of all sets

$$\langle \alpha, U \rangle = \{ \langle \alpha * \delta \rangle \mid \delta : ([0, 1], 0) \rightarrow (U, \alpha(1)) \}$$

with U open in X , and $\alpha(1) \in U$.

We call \tilde{X}_H the *path space of X with respect to H* , and $p_H : (\tilde{X}_H, \tilde{x}_0) \rightarrow (X, x_0)$, given by $p_H(\langle \alpha \rangle) = \alpha(1)$ the *endpoint projection with respect to H* .

The path space \tilde{X}_H is always a connected, locally path-connected space. The endpoint projection p_H is always a continuous surjection. It is an open map if and only if X is locally path-connected. Moreover, lifts always exist for p_H as discussed in [10] and [9]: Suppose Y is connected and locally path-connected, $f : Y \rightarrow X$ a continuous map, $y \in Y$ and $\langle \alpha \rangle \in \tilde{X}_H$ with $p_H(\langle \alpha \rangle) = f(y)$. Then there is a continuous lift $\tilde{f} : (Y, y) \rightarrow (\tilde{X}_H, \langle \alpha \rangle)$ such that $p_H \circ \tilde{f} = f$ provided $f_{\#}(\pi_1(Y, y)) \leq [\alpha^-]H[\alpha]$. For example, we may define $\tilde{f}(z) = \langle \alpha * (f \circ \tau) \rangle$, where $\tau : [0, 1] \rightarrow Y$ is any path from $\tau(0) = y$ to $\tau(1) = z$. Note that $[\alpha^-]H[\alpha] \leq p_{H\#}(\pi_1(\tilde{X}_H, \langle \alpha \rangle))$. Moreover, if $p_H : \tilde{X}_H \rightarrow X$ has unique path lifting, then $p_{H\#} : \pi_1(\tilde{X}_H, \langle \alpha \rangle) \rightarrow \pi_1(X, f(y))$ is a monomorphism onto $[\alpha^-]H[\alpha]$.

However these lifts may not be unique. (See below.) When paths lift uniquely in the endpoint projection, then the endpoint projection is a generalized covering projection. Finally, if $p : E \rightarrow B$ is a generalized covering projection, then it is equivalent to the endpoint projection with respect to $H = p_{\#}(\pi_1(E, e_0))$ [2]. This allows for a straightforward procedure for working with the generalized covering projection with respect to any given subgroup of the fundamental group; construct the path space and the endpoint projection with respect to the given subgroup, and test the endpoint projection for the unique path lifting property.

Many standard features of covering spaces are preserved by this generalization. For instance, let $p_H : (\tilde{X}_H, \tilde{x}_0) \rightarrow (X, x_0)$ be a generalized covering projection as defined above, where for simplicity $\tilde{X} = \tilde{X}_H$ and $p_H = p$. For example, the usual arguments show that $\text{Aut}(\tilde{X} \xrightarrow{p} X) \approx \mathbb{N}_G(H)/H$

where $G = \pi_1(X, x_0)$ and $\mathbb{N}_G(H) = \{g \in G : gHg^{-1} = H\}$. We see that $\mathbb{N}_G(H)$ acts on \tilde{X} from the left ($\mathbb{N}_G(H) \curvearrowright \tilde{X}$), by $[\alpha].\langle\beta\rangle = \langle\alpha * \beta\rangle$. (We see that when H is trivial the left action reduces to $[\alpha].[\beta] = [\alpha * \beta]$.) In particular, $\mathbb{N}_G(H)$ acts on $p^{-1}(x)$ from the left ($\mathbb{N}_G(H) \curvearrowright p^{-1}(x)$) for any $x \in X$.

An element $h \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$ is fully determined by its restriction $h|_{p^{-1}(x)} : p^{-1}(x) \rightarrow p^{-1}(x)$. Given $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$, there exists $h \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$ with $h(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_{\#}(\pi_1(\tilde{X}, \tilde{x}_1)) = p_{\#}(\pi_1(\tilde{X}, \tilde{x}_2))$.

Given $y \in X$ and $\beta : [0, 1] \rightarrow X$ with $\beta(0) = y$, put $\tilde{y} \in p^{-1}(y)$. If we express \tilde{y} as $\langle\alpha\rangle$, then by construction of the monodromy, we have $\phi_{\beta}(\tilde{y}) = \langle\alpha * \beta\rangle$. Therefore, G acts on $p^{-1}(x)$ from the right, by $\langle\alpha\rangle.[\beta] = \langle\alpha\beta\rangle$, ($p^{-1}(x) \curvearrowright G$).

For any $h \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$, $\tilde{y} \in p^{-1}(x)$, $g \in G$, we have $h(\tilde{y}.g) = h(\tilde{y}).g$ by definition of lift.

The left action of $\mathbb{N}_G(H)$ on $p^{-1}(x)$ induces a surjective homomorphism $\psi : \mathbb{N}_G(H) \rightarrow \text{Aut}(\tilde{X} \xrightarrow{p} X)$ with kernel H : For $[\alpha] \in \mathbb{N}_G(H)$, we have $p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0)) = H = [\alpha]^{-1}H[\alpha] = p_{\#}(\pi_1(\tilde{X}, \langle\alpha\rangle))$, so that there exists a unique $h \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$ with $h(\tilde{x}) = \langle\alpha\rangle$. Define $\psi([\alpha]) = h$. Note that for $\tilde{y} = \langle\beta\rangle \in p^{-1}(x)$, we have

$$h(\tilde{y}) = h(\tilde{x}.\langle\beta\rangle) = h(\tilde{x}).[\beta] = \langle\alpha\rangle.[\beta] = \langle\alpha * \beta\rangle = [\alpha].\langle\beta\rangle = [\alpha].\tilde{y},$$

i.e., $\psi([\alpha])(\langle\beta\rangle) = [\alpha].\langle\beta\rangle$ and ψ induces an isomorphism $\mathbb{N}_G(H)/H \approx \text{Aut}(\tilde{X} \xrightarrow{p} X)$.

In particular, if $H = 1$, we have an isomorphism $G \approx \text{Aut}(\tilde{X} \xrightarrow{p} X)$ that is induced by the left action of G on $p^{-1}(x)$.

The discussion above shows that on a generalized covering projection, the monodromy $\phi_{\beta}(\langle\alpha\rangle)$ is equivalent to the right action $\langle\alpha * \beta\rangle = \langle\alpha\rangle.[\beta]$. This provides a well defined extension of the monodromy to the endpoint projection, where paths may not lift uniquely:

Definition 2.3 (General Monodromy). Let $p_H : \tilde{X}_H \rightarrow X$ be the endpoint projection, $y, z \in X$, and $\beta : [0, 1] \rightarrow X$ a path that starts at y and ends at z . For every $\langle\alpha\rangle \in p_H^{-1}(y)$, define the *monodromy* $\phi_{\beta} : p^{-1}(y) \rightarrow p^{-1}(z)$ by $\phi_{\beta}(\langle\alpha\rangle) = \langle\alpha * \beta\rangle$.

The following definition appears first in [1] and [16], was given its name in [6], and was then extended to a relative version in [9]:

Definition 2.4 (Homotopically Hausdorff with respect to H). Let X be a path-connected topological space, $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. We call X *homotopically Hausdorff with respect to H* if, for every $x \in X$, for every path $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = x_0$ and $\alpha(1) = x$, and for every $g \in \pi_1(X, x_0) \setminus H$, there exists a neighborhood U of x such that for all $\delta : ([0, 1], \{0, 1\}) \rightarrow (U, x)$, $[\alpha * \delta * \alpha^-] \notin Hg$. If this condition is satisfied for the trivial subgroup of $\pi_1(X, x_0)$, then we say X is *homotopically Hausdorff*.

The following observation from [9] describes the relationship between homotopically Hausdorff and Hausdorff.

Remark 2.5. Suppose X is Hausdorff, $H \leq \pi_1(X, x_0)$ and consider the endpoint projection $p_H : \tilde{X}_H \rightarrow X$. Then \tilde{X}_H is Hausdorff if and only if X is homotopically Hausdorff.

The following definition originates in [16] and was first defined in its current version in [8], with the relative version first appearing in [3]:

Definition 2.6 (Homotopically Path Hausdorff with respect to H). Let X be a path-connected topological space, $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. We call X *homotopically path Hausdorff with respect to H* if, for every pair of paths $\alpha, \beta : [0, 1] \rightarrow X$ with $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1)$, and $[\alpha * \beta^-] \notin H$, there is a partition $0 = t_0 < t_1 < \dots < t_n = 1$ and a sequence of open subsets U_1, U_2, \dots, U_n with $\alpha([t_{i-1}, t_i]) \subseteq U_i$ such that if $\gamma : [0, 1] \rightarrow X$ is another path satisfying $\gamma([t_{i-1}, t_i]) \subseteq U_i$ for $1 \leq i \leq n$ and $\gamma(t_i) = \alpha(t_i)$ for every $0 \leq i \leq n$, then $[\gamma * \beta^-] \notin H$. If this condition is satisfied for the trivial subgroup of $\pi_1(X, x_0)$, then we say X is *homotopically path Hausdorff*.

These two notions give necessary and sufficient conditions for the endpoint projection $p_H : \tilde{X}_H \rightarrow X$ to be a generalized covering projection. The first, homotopically Hausdorff with respect to H , is a necessary condition [9], while the second, homotopically path Hausdorff with respect to H , is a sufficient condition [3]. In particular, when a space X is homotopically path Hausdorff with respect to H , the endpoint projection has unique path lifting.

In [10], the definition for X to be homotopically Hausdorff with respect to H is stated as a property on fibers of the endpoint projection with respect to H . We show these definitions are equivalent:

Lemma 2.7. Suppose X is path-connected, $H \leq \pi_1(X, x_0)$ and let $p_H : (\tilde{X}_H, \tilde{x}_0) \rightarrow (X, x_0)$ be the endpoint projection with respect to H . Then X is homotopically Hausdorff with respect to H if and only if $p_H^{-1}(x)$ is T_1 for every $x \in X$.

Proof. Suppose X is homotopically Hausdorff with respect to H . Let $x \in X$ be arbitrary and suppose $\tilde{x}_1, \tilde{x}_2 \in p_H^{-1}(x)$ with $\tilde{x}_1 \neq \tilde{x}_2$. So, if α and β are paths in X such that $\tilde{x}_1 = \langle \alpha \rangle$ and $\tilde{x}_2 = \langle \beta \rangle$, then $[\beta * \alpha^-] \notin H$. Put $g = [\beta * \alpha^-]$. By assumption, there exists an open neighborhood U of x in X such that for every closed path $\delta : [0, 1] \rightarrow U$ with $\delta(0) = x$, $[\alpha * \delta * \alpha^-] \notin Hg$. Suppose there exists $\langle \alpha * \delta \rangle \in \langle \alpha, U \rangle$ with $\langle \alpha * \delta \rangle = \langle \beta \rangle$. Then $[\alpha * \delta * \beta^-] \in H$, so that $[\alpha * \delta * \alpha^-] \in H[\beta * \alpha^-] = Hg$, a contradiction. Consequently, $p_H^{-1}(x) \cap \langle \alpha, U \rangle$ is a neighborhood of $\langle \alpha \rangle$ in $p_H^{-1}(x)$ which does not contain $\langle \beta \rangle$.

Now suppose $p_H^{-1}(x)$ is T_1 for every $x \in X$. Let $x \in X$ be arbitrary and let α be a path from x_0 to x , $g \in \pi_1(X, x_0) \setminus H$. Suppose $g = [\beta]$. Then $\langle \alpha \rangle \neq \langle \beta * \alpha \rangle \in p_H^{-1}(x)$. Since $p_H^{-1}(x)$ is T_1 by assumption, there exists a neighborhood U of x such that $\langle \beta * \alpha \rangle \notin \langle \alpha, U \rangle$. Suppose that there is a loop δ in U with $\delta(0) = x$ such that $[\alpha * \delta * \alpha^-] \in Hg$. Then $\langle \alpha * \delta * \alpha^- \rangle = \langle \beta \rangle$ so that $\langle \beta * \alpha \rangle = \langle \alpha * \delta \rangle \in \langle \alpha, U \rangle$, a contradiction. \square

The Griffith Twin Cone is a well know example which can be found in [11]. It consists of joining two copies of a cone over the Hawaiian Earring at the distinguished points of their bases. Together with the Hawaiian Earring, these examples are notable in how they show the relationship between the properties we have discussed so far: The Hawaiian Earring has no universal covering projection in the classical sense. However it does have a simply connected, and therefore universal, generalized covering projection that has exactly one non-discrete fiber [9]. Thus, the Hawaiian Earring is homotopically Hausdorff. However, the Griffiths Twin Cone is not homotopically Hausdorff, since the alternating loop $l_1 * l'_1 * l_2 * l'_2 * \dots$ between the bases is not homotopic to the constant path, although it can be homotoped arbitrarily close to the join point [11].

Remark 2.8. An example of a space X for which the endpoint projection $p_H : \tilde{X}_H \rightarrow X$ has unique path lifting with $H = 1$, but which is not homotopically path Hausdorff, is space B of [7]; see [4, Remark 6.10]. An example of space X which is homotopically Hausdorff, but for which the endpoint projection $p_H : \tilde{X} \rightarrow X$ does not have unique path lifting for $H = 1$ can be found in [15]. For further examples that are homotopically Hausdorff relative to nontrivial subgroups and nontrivial normal subgroups, but whose corresponding endpoint projections do not have unique path lifting, see [4, Examples 5.1 and 5.2].

We close this chapter with two short observations:

Lemma 2.9. *Let X be a path-connected space, $x_0 \in X$, $H \leq \pi_1(X, x_0)$. Consider the endpoint projection $p_H : \tilde{X} \rightarrow X$. Suppose β is a path in X from x to y such that the monodromy $\phi_\beta : p_H^{-1}(x) \rightarrow p_H^{-1}(y)$ is continuous. If $p_H^{-1}(y)$ is T_1 , then so is $p_H^{-1}(x)$.*

Proof. Since $\phi_\beta : p_H^{-1}(x) \rightarrow p_H^{-1}(y)$ is a continuous bijection, the inverse images of closed singletons are closed singletons. \square

Lemma 2.10. *Let X be a path-connected space, $x_0, x, y, z \in X$, and $H \leq \pi_1(X, x_0)$. Consider the endpoint projection $p_H : \tilde{X}_H \rightarrow X$. Suppose γ is a path from x_0 to x , β a path from x to y , δ a path from y to z and $\alpha = \beta * \delta$. Let U and V be open subsets of X such that $\text{Im}(\delta) \subseteq V$. Then $\phi_{\alpha-}(\langle \gamma * \alpha, V \rangle \cap p_H^{-1}(\alpha(1))) \subseteq \langle \gamma, U \rangle \cap p_H^{-1}(\alpha(0))$ if and only if $\phi_{\beta-}(\langle \gamma * \beta, V \rangle \cap p_H^{-1}(\beta(1))) \subseteq \langle \gamma, U \rangle \cap p_H^{-1}(\alpha(0))$.*

Proof. Suppose that $\phi_{\alpha-}(\langle \gamma * \alpha, V \rangle \cap p_H^{-1}(\alpha(1))) \subseteq \langle \gamma, U \rangle \cap p_H^{-1}(\alpha(0))$. Let $\langle \eta \rangle \in \langle \gamma * \beta, V \rangle \cap p_H^{-1}(\beta(1))$. Then $\langle \eta \rangle = \langle \gamma * \beta * \tau \rangle$ for some loop τ in V with $\tau(0) = y$. Hence $\phi_{\beta-}(\langle \eta \rangle) = \langle \gamma * \beta * \tau * \beta^- \rangle = \langle (\gamma * \beta * \delta) * (\delta^- * \tau * \delta) * (\delta^- * \beta^-) \rangle = \phi_{\alpha-}(\langle \gamma * \alpha * (\delta^- * \tau * \delta) \rangle) \in \langle \gamma, U \rangle \cap p_H^{-1}(\alpha(0))$.

The other implication is shown similarly. \square

Chapter 3

Small Loop Transfer and Related Extensions

In [5], the concept of a small loop transfer space was introduced:

Definition 3.1 (Small Loop Transfer). A path-connected topological space X is a *small loop transfer space*, or SLT space, if for every path α in X and every neighborhood U of $x_0 = \alpha(0)$ there is a neighborhood V of $x_1 = \alpha(1)$ such that given a loop $\beta : (S^1, 1) \rightarrow (V, x_1)$ there is a loop $\gamma : (S^1, 1) \rightarrow (U, x_0)$ that is homotopic to $\alpha * \beta * \alpha^{-1}$ rel. x_0 .

The Hawaiian Earring is not a SLT space, as shown in Proposition 4.10 of [5]. From this paper we also quote the following fact:

Proposition 3.2. *X is a small loop transfer space if and only if every path α in X from x_0 to x_1 induces a homeomorphism $h_\alpha : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$, given by $h_\alpha([\beta]) = [\alpha * \beta * \alpha^{-1}]$, where both $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are equipped with the whisker topology: construct the endpoint projections $p : \tilde{X}_0 \rightarrow X$ with respect to $1 \leq \pi_1(X, x_0)$ and $q : \tilde{X}_1 \rightarrow X$ with respect to $1 \leq \pi_1(X, x_1)$, then apply the subspace topology inherited by $\pi_1(X, x_0) \subseteq \tilde{X}_0$ and $\pi_1(X, x_1) \subseteq \tilde{X}_1$.*

It is clear from the construction of the monodromy and the definition of the whisker topology on $\pi_1(X, x_0)$ and on $\pi_1(X, x_1)$ that this proposition is equivalent to the following:

Corollary 3.3. *Let $p : \tilde{X} \rightarrow X$ be the endpoint projection with respect to the trivial subgroup of $\pi_1(X, x)$. Then X is a small loop transfer space if and only if for every path α in X from x_0 to x_1 , the monodromy $\phi_\alpha : p^{-1}(x_0) \rightarrow p^{-1}(x_1)$ is continuous.*

In 2017, Pashaei et. al. [13] extended the concept of a small loop transfer space to endpoint projections p_H in order to accommodate for arbitrary subgroups of the fundamental group.

Definition 3.4 (H -SLT Space at x_0). Let X be a path-connected topological space and let $H \leq \pi_1(X, x_0)$. We say that the topological space X is an H -SLT space at x_0 , if, for every path α beginning at x_0 , and for every open subset U with $x_0 \in U$, there is an open subset V with $\alpha(1) \in V$ such that for every loop β in V based at $\alpha(1)$ there is a loop γ in U based at x_0 such that $[\alpha * \beta * \alpha^- * \gamma] \in H$.

Definition 3.5 (H -SLT Space). Let $H \leq \pi_1(X, x_0)$. Then X is called an H -SLT space if for every $x \in X$ and for every path δ from x_0 to x , X is a $[\delta^-]H[\delta]$ -SLT at x .

In the following lemmas and subsequent corollary, we show some relationships between H -SLT spaces and continuous monodromy between fibers:

Lemma 3.6. *Let X be a path-connected space, let $H \leq \pi_1(X, x_0)$, and consider the endpoint projection $p_H : \tilde{X}_H \rightarrow X$. The following are equivalent:*

1. *For every pair of points y, z in X and every path β with $\beta(0) = y$ and $\beta(1) = z$, the monodromy $\phi_\beta : p_H^{-1}(y) \rightarrow p_H^{-1}(z)$ is continuous.*
2. *X is a H -SLT space.*

Proof. Assume $\phi_\beta : p_H^{-1}(\beta(0)) \rightarrow p_H^{-1}(\beta(1))$ is continuous for every path β in X . Let $x \in X$ be arbitrary, let $\delta : [0, 1] \rightarrow X$ be a path from x_0 to x , and let $\alpha : [0, 1] \rightarrow X$ be any path with $\alpha(0) = x$. Let U be an open neighborhood of x , and so by assumption, $\phi_{\alpha^-}^{-1}(p_H^{-1}(x) \cap \langle \delta, U \rangle)$ is open in $p_H^{-1}(x)$, thus there exists a basis element $p_H^{-1}(\alpha(1)) \cap \langle \delta * \alpha, V \rangle \subseteq \phi_{\alpha^-}^{-1}(p_H^{-1}(x) \cap \langle \delta, U \rangle)$. Now let β be a closed path contained in V such that $\beta(0) = \alpha(1)$. Then $\langle \delta * \alpha * \beta \rangle \in p_H^{-1}(\alpha(1)) \cap \langle \delta * \alpha, V \rangle$, and so there exists γ a closed path contained in U with $\gamma(0) = x$ such that $\langle \delta * \gamma \rangle = \phi_{\alpha^-}(\langle \delta * \alpha * \beta \rangle) = \langle \delta * \alpha * \beta * \alpha^- \rangle$. Therefore $[\delta * \alpha * \beta * \alpha^- * \gamma^- * \delta^-] \in H$ so that $[\alpha * \beta * \alpha^- * \gamma^-] \in [\delta^-]H[\delta]$, that is X is a $[\delta^-]H[\delta]$ -SLT space at x .

Now assume X is a H -SLT space, let $x, y \in X$ be arbitrary with $\alpha : [0, 1] \rightarrow X$ a path from x to y . We will show that $\phi_{\alpha^-} : p_H^{-1}(y) \rightarrow p_H^{-1}(x)$ is continuous. Let $\langle \psi \rangle \in p_H^{-1}(y)$, and put $\delta = \psi * \alpha^-$. Then $\phi_{\alpha^-}(\langle \psi \rangle) = \langle \delta \rangle$. Let U be an open neighborhood of x . Since X is an H -SLT space, there is an open neighborhood V of y such that for every closed path β in V with $\beta(0) = y$ there is a closed path γ in U with $\gamma(0) = x$ such that $[\alpha * \beta * \alpha^- * \gamma^-] \in [\delta^-]H[\delta]$. We show that $\phi_{\alpha^-}(p_H^{-1}(y) \cap \langle \psi, V \rangle) \subseteq p_H^{-1}(x) \cap \langle \delta, U \rangle$. To this end, let β be a closed path in V with $\beta(0) = y$, and let γ be as above. Then $\phi_{\alpha^-}(\langle \psi * \beta \rangle) = \langle \psi * \beta * \alpha^- \rangle = \langle \psi * \alpha^- * \alpha * \beta * \alpha^- \rangle = \langle \delta * \alpha * \beta * \alpha^- \rangle = \langle \delta * \gamma \rangle$, because $[\delta * \alpha * \beta * \alpha^- * \gamma^- * \delta^-] \in H$. \square

Lemma 3.7. *Let X be a path-connected topological space. Let $H \leq \pi_1(X, x_0)$ and consider the endpoint projection $p_H : \tilde{X}_H \rightarrow X$. Then the following are equivalent:*

1. *For every path $\beta : [0, 1] \rightarrow X$ with $\beta(1) = x_0$, the monodromy $\phi_\beta : p_H^{-1}(\beta(0)) \rightarrow p_H^{-1}(x_0)$ is continuous.*
2. *For every loop δ at x_0 , X is a $[\delta^-]H[\delta]$ -SLT space at x_0 .*

Proof. In the proof of Lemma 3.6, put $x = x_0$. \square

Corollary 3.8. *Let X be a path-connected space and $H \trianglelefteq \pi_1(X, x_0)$ be a normal subgroup and consider $p_H : \tilde{X}_H \rightarrow X$. The following are equivalent:*

1. *The monodromy $\phi_\beta : p_H^{-1}(\beta(0)) \rightarrow p_H^{-1}(x_0)$ is continuous for every $\beta : [0, 1] \rightarrow X$ a path with $\beta(1) = x_0$,*
2. *X is a H -SLT space at x_0*

Chapter 4

Extensions and Generalizations

We begin with [13, Theorem 2.4], a theorem that connects a necessary condition for the endpoint projection to be a generalized covering projection with a sufficient condition.

Theorem 4.1 (Pashaei, Mashayekhy, Torabi, Abdullahi Rashid). *Let X be a path-connected space, let $H \trianglelefteq \pi_1(X, x_0)$ be a normal subgroup and X be an H -SLT space at x_0 . Then X is homotopically path Hausdorff with respect to H if and only if X is homotopically Hausdorff with respect to H .*

As stated in Chapter 2, if X is homotopically path Hausdorff with respect to H , then X is homotopically Hausdorff with respect to H . We present three extensions of this result.

Theorem 4.2 (Extension 1). *Let X be a path-connected space, let $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. Suppose that X is homotopically Hausdorff with respect to H and that for every $x_1 \in X$ and for every path β in X from x_0 to x_1 , the monodromy $\phi_{\beta^-} : p_H^{-1}(x_1) \rightarrow p_H^{-1}(x_0)$ is continuous. Then the endpoint projection $p_H : \tilde{X}_H \rightarrow X$ has unique path lifting.*

Before providing the proof for this theorem, it will be useful to prove the following two lemmas:

Lemma 4.3. *Let X be path-connected space, let $p_H : \tilde{X}_H \rightarrow X$ be the endpoint projection, and let $f, g : [0, 1] \rightarrow \tilde{X}_H$ be two continuous lifts of a path $\beta : [0, 1] \rightarrow X$ with $p_H \circ f = \beta = p_H \circ g$ and $f(0) = g(0)$. Suppose that the monodromy $\phi_{\beta_t^-} : p_H^{-1}(\beta(t)) \rightarrow p_H^{-1}(\beta(0))$ is continuous for every $t \in [0, 1]$, where $\beta_t^-(s) = \beta^-(st)$. Express g as $g(t) = \langle \alpha_t \rangle$ and suppose that f is the standard lift with $f(t) = \langle \alpha_0 * \beta_t \rangle$. Then $t \mapsto \langle \alpha_t * \beta_t^- \rangle$ defines a continuous path $h : [0, 1] \rightarrow p_H^{-1}(\beta(0))$.*

Proof. Note that $h(t) \in p_H^{-1}(\beta(0))$ for all $t \in [0, 1]$, since $\alpha_t * \beta_t^-(1) = \beta(0)$. Let γ be a path from x_0 to $\beta(0)$, and suppose U is an open neighborhood of $\beta(0)$. Then $\langle \gamma, U \rangle \cap p_H^{-1}(\beta(0))$ is an arbitrary basic open set of $p_H^{-1}(\beta(0))$. Suppose $t \in h^{-1}(\langle \gamma, U \rangle \cap p_H^{-1}(\beta(0)))$. Then $\phi_{\beta_t^-}(\langle \alpha_t \rangle) = \langle \alpha_t * \beta_t^- \rangle \in \langle \gamma, U \rangle$, and so $\langle \alpha_t * \beta_t^-, U \rangle = \langle \gamma, U \rangle$. By the continuity of $\phi_{\beta_t^-}$, there exists an open neighborhood V of $\alpha_t(1) = \alpha(t) = \beta(t)$ such that $\phi_{\beta_t^-}(\langle \alpha_t \rangle) \in \phi_{\beta_t^-}(\langle \alpha_t, V \rangle \cap p_H^{-1}(\beta(t))) \subseteq \langle \alpha_t * \beta_t^-, U \rangle \cap p_H^{-1}(\beta(0))$. Furthermore, by the continuity of β and g , there exists an interval I open in $[0, 1]$ with $t \in I$ such that $\beta(I) \subseteq V$ and $g(I) \subseteq \langle \alpha_t, V \rangle$. Now let $s \in I$. Then $g(s) = \langle \alpha_s \rangle = \langle \alpha_t * \delta_1 \rangle$ where $\text{Im}(\delta_1) \subseteq V$, and $f(s) = \langle \alpha_0 * \beta_s \rangle = \langle \alpha_0 * \beta_t * \delta_2 \rangle$ where $\beta_s = \beta_t * \delta_2$ with $\text{Im}(\delta_2) \subseteq \beta(I) \subseteq V$. Thus $\langle \alpha_s * \beta_s^- \rangle = \langle \alpha_s * \delta_1 * \delta_2^- * \beta_t^- \rangle = \phi_{\beta_t^-}(\langle \alpha_t * \delta_1 * \delta_2^- \rangle) \in \langle \alpha_t * \beta_t^-, U \rangle = \langle \gamma, U \rangle$, and so $t \in I \subseteq h^{-1}(\langle \gamma, U \rangle \cap p_H^{-1}(\beta(0)))$. \square

Lemma 4.4. *Let X be a path-connected space. Let $x \in X$, $H \leq \pi_1(X, x_0)$, and $p_H : \tilde{X}_H \rightarrow X$ the endpoint projection. If the fiber $p_H^{-1}(x)$ is T_1 , then it is totally disconnected.*

Proof. Let $\langle \alpha \rangle$ and $\langle \beta \rangle$ be distinct elements of $p_H^{-1}(x)$. Since the fiber is T_1 , there exists an open set U such that $\langle \beta \rangle \notin \langle \alpha, U \rangle \cap p_H^{-1}(x)$. Put $W_1 = \langle \alpha, U \rangle \cap p_H^{-1}(x)$ and $W_2 = \bigcup \{ \langle \gamma, U \rangle \cap p_H^{-1}(x) \mid \langle \gamma \rangle \neq \langle \alpha \rangle \}$. Then $\{W_1, W_2\}$ is a separation of $p_H^{-1}(x)$ into disjoint open sets with $\langle \alpha \rangle \in W_1$ and $\langle \beta \rangle \in W_2$. \square

We now provide a proof of Theorem 4.2 using these two lemmas, the main idea behind the proof being that continuous functions from connected spaces into totally disconnected spaces are constant.

Proof of Theorem 4.2. Let $f, g : [0, 1] \rightarrow \tilde{X}_H$ be two continuous lifts of a path $\beta : [0, 1] \rightarrow X$ with $p_H \circ f = \beta = p_H \circ g$ and $f(0) = g(0)$. Without loss of generality, we may say that f is the standard lift of β given by $f(t) = \langle \alpha_0 * \beta_t \rangle$ and that g is some other lift of β , say $g(t) = \langle \alpha_t \rangle$. By Lemma 4.3, the function $h : [0, 1] \rightarrow p_H^{-1}(\beta(0))$ via $h(t) = \langle \alpha_t * \beta_t^- \rangle$ is continuous and thus, since $p_H^{-1}(\beta(0))$ is totally disconnected by Lemma 4.4, h must be a constant function. Therefore, for any t , $\langle \alpha_0 \rangle = \langle \alpha_0 * \beta_0^- \rangle = \langle \alpha_t * \beta_t^- \rangle$, which gives $\langle \alpha_0 * \beta_t \rangle = \langle \alpha_t * \beta_t^- * \beta_t \rangle$ so that $f(t) = \langle \alpha_0 * \beta_t \rangle = \langle \alpha_t \rangle = g(t)$. \square

We now introduce a new concept that will turn out to be useful:

Definition 4.5 (Locally Quasinormal). Let X be a path-connected space, $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. We say that H is *locally quasinormal* if, for every $x \in X$, for every path α from x_0 to x ,

and for every neighborhood V of x , there exists a neighborhood U of x with $x \in U \subseteq V$ such that

$$H\pi(\alpha, U) = \pi(\alpha, U)H,$$

where $\pi(\alpha, U) = \{[\alpha * \delta * \alpha^-] \mid \text{Im}(\delta) \subseteq U, \delta(0) = \delta(1) = x\}$.

Note that while the subgroups $\pi(\mathcal{U}, x_0) \trianglelefteq \pi_1(X, x_0)$ are normal, the subgroups $\pi_1(\alpha, U)$ are typically not normal.

Recall the following from elementary group theory:

Remark 4.6. For subgroups H and K of a group G , the following are equivalent:

1. $KH \subseteq HK$
2. $KH = HK$
3. HK is a subgroup of G

Proof. (1) \Rightarrow (3). Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. We show $(h_1k_1)(h_2k_2)^{-1} \in HK$. Now, $(h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$ and since $(k_2k_2^{-1})h_2^{-1} \in KH \subseteq HK$, we have $(k_2k_2^{-1})h_2^{-1} = h_3k_3$ for some $k_3 \in K, h_3 \in H$. Then $(h_1k_1)(h_2k_2)^{-1} = h_1h_3k_3 \in HK$ as desired.

(3) \Rightarrow (2). Since $K \leq HK$ and $H \leq HK$, and HK is a subgroup, $KH \subseteq HK$. For the reverse inclusion, let $h \in H, k \in K$. Then $k^{-1}h^{-1} = (hk)^{-1} \in HK$. So, $k^{-1}h^{-1} = \tilde{h}\tilde{k}$ for some $\tilde{h} \in H$ and $\tilde{k} \in K$. Hence $hk = \tilde{k}^{-1}\tilde{h}^{-1} \in KH$. \square

We now provide an alternate statement for what it means to be locally quasinormal, by showing the local quasinormal condition is equivalent to the property that for any point in the fiber, there is an arbitrarily small basis set such that the basis set is fixed by certain monodromies within the fiber.

Proposition 4.7. *Let X be a path-connected space, $x_0 \in X$, $H \leq \pi_1(X, x_0)$. Consider the endpoint projection $p_H : \tilde{X}_H \rightarrow X$. Let $x \in X$, α a path from x_0 to x , and U a neighborhood of x . The following are equivalent:*

1. $H\pi(\alpha, U) = \pi(\alpha, U)H$

2. $\phi_\beta(\langle \alpha, U \rangle \cap p_H^{-1}(x)) = \langle \alpha, U \rangle \cap p_H^{-1}(x)$ for all $[\beta] \in [\alpha^-]H[\alpha]$

Note that $\phi_\beta(\langle \alpha \rangle) = \langle \alpha \rangle$ if and only if $[\beta] \in [\alpha^-]H[\alpha]$.

Proof. (1) \Rightarrow (2). Let $[\beta] \in [\alpha^-]H[\alpha]$. Then $[\beta] = [\alpha^- * \gamma * \alpha]$ for some $[\gamma] \in H$. We show that $\phi_\beta(\langle \alpha, U \rangle \cap p_H^{-1}(x)) \subseteq \langle \alpha, U \rangle \cap p_H^{-1}(x)$. Let $\langle \psi \rangle \in \langle \alpha, U \rangle \cap p_H^{-1}(x)$. Then $\langle \psi \rangle = \langle \alpha * \delta \rangle$ for some loop δ in U with $\delta(0) = x$. Since $[\alpha * \delta * \alpha^-][\gamma] \in \pi(\alpha, U)H \subseteq H\pi(\alpha, U)$, there is a $[\tilde{\gamma}] \in H$ and a loop $\tilde{\delta}$ in U with $\tilde{\delta}(0) = x$ so that $[\alpha * \delta * \alpha^-][\gamma] = [\tilde{\gamma}][\alpha * \tilde{\delta} * \alpha^-]$. Hence, $[\alpha * \delta * \alpha^- * \gamma * \alpha * \tilde{\delta}^- * \alpha^-] = [\tilde{\gamma}] \in H$, so that $\phi_\beta(\langle \psi \rangle) = \langle \psi * \beta \rangle = \langle \alpha * \delta * \alpha^- * \gamma * \alpha \rangle = \langle \alpha * \tilde{\delta} \rangle \in \langle \alpha, U \rangle \cap p_H^{-1}(x)$. Therefore, $\phi_\beta(\langle \alpha, U \rangle \cap p_H^{-1}(x)) \subseteq \langle \alpha, U \rangle \cap p_H^{-1}(x)$.

The reverse inclusion follows from applying the argument above to $[\beta^-] \in [\alpha^-]H[\alpha]$ instead of $[\beta]$, since $\phi_{\beta^-} = \phi_\beta^{-1}$.

(2) \Rightarrow (1). We show that $\pi(\alpha, U)H \subseteq H\pi(\alpha, U)$. Let $[\theta] \in \pi(\alpha, U)H$. Then $[\theta] = [\alpha * \delta * \alpha^- * \gamma]$ for some loop δ in U with $\delta(0) = x$ and $[\gamma] \in H$. Put $\beta = \alpha^- * \gamma * \alpha$. Then $[\beta] \in [\alpha^-]H[\alpha]$ and $\langle \alpha * \delta \rangle \in \langle \alpha, U \rangle \cap p_H^{-1}(x)$. By assumption, $\langle \alpha * \delta * \alpha^- * \gamma * \alpha \rangle = \langle \alpha * \delta * \beta \rangle = \phi_\beta(\langle \alpha * \delta \rangle) \in \langle \alpha, U \rangle \cap p_H^{-1}(x)$. So, $\langle \alpha * \delta * \alpha^- * \gamma * \alpha \rangle = \langle \alpha * \tilde{\delta} \rangle$ for some loop $\tilde{\delta}$ in U with $\tilde{\delta}(0) = x$. That is, $[\alpha * \delta * \alpha^- * \gamma * \alpha * \tilde{\delta}^- * \alpha^-] \in H$, so that $[\alpha * \delta * \alpha^- * \gamma * \alpha * \tilde{\delta}^- * \alpha^-] = [\tilde{\gamma}]$ for some $[\tilde{\gamma}] \in H$. Hence $[\theta] = [\alpha * \delta * \alpha^- * \gamma] = [\tilde{\gamma} * \alpha * \tilde{\delta} * \alpha^-] \in H\pi(\alpha, U)$.

For the reverse inclusion, refer to Remark 4.6. □

In particular, we see that H is locally quasinormal whenever H is a normal subgroup of the fundamental group by the definition of locally quasinormal, and the equivalent interpretation shows that H is locally quasinormal whenever $p_H^{-1}(x)$ is discrete for every x .

An example which illustrates the difference between H a normal subgroup and H a locally quasinormal subgroup can be found in [10]. In this example X is the Hawaiian Earring, $H \neq 1$, $p_H : \tilde{X}_H \rightarrow X$ is a local homeomorphism so that the fibers $p_H^{-1}(x)$ are discrete for every x , and H does not contain any nontrivial normal subgroup of $\pi_1(X, x_0)$.

Theorem 4.8 (Extension 2). *Let X be a path-connected space, $x_0 \in X$, and H a locally quasinormal subgroup of $\pi_1(X, x_0)$. Consider the endpoint projection $p_H : \tilde{X}_H \rightarrow X$. Suppose that X is homotopically Hausdorff with respect to H and that, for every path $\beta : [0, 1] \rightarrow X$ with $\beta(1) = x_0$, the monodromy $\phi_\beta : p_H^{-1}(\beta(0)) \rightarrow p_H^{-1}(x_0)$ is continuous. Then X is homotopically path Hausdorff with respect to H .*

Proof. Let $\alpha, \beta : [0, 1] \rightarrow X$ be two paths with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1)$ such that $[\alpha * \beta^-] \notin H$. Let c denote the constant path at x_0 . Then $\langle c \rangle$ and $\langle \beta * \alpha^- \rangle$ are distinct elements of the fiber $p_H^{-1}(x_0)$. Hence, there is an open subset $U \subseteq X$ with $x_0 \in U$ such that $\langle \beta * \alpha^- \rangle \notin \langle c, U \rangle \cap p_H^{-1}(x_0)$. We may assume $\pi(c, U)H = H\pi(c, U)$. For each $t \in [0, 1]$, define $\alpha_t(s) = \alpha(st)$. By assumption, each monodromy $\phi_{\alpha_t^-} : p_H^{-1}(\alpha(t)) \rightarrow p_H^{-1}(x_0)$ is continuous. Hence, for each $t \in [0, 1]$, there is an open subset V_t of X with $\alpha(t) \in V_t$ such that $\phi_{\alpha_t^-}(\langle \alpha_t, V_t \rangle \cap p_H^{-1}(\alpha(t))) \subseteq \langle c, U \rangle \cap p_H^{-1}(x_0)$. Using compactness of $[0, 1]$ and Lemma 2.10, we find a finite subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ such that, for every $1 \leq i \leq n-1$, $\alpha([t_i, t_{i+1}]) \subseteq V_{t_i}$ and $\alpha([t_0, t_1]) \subseteq U$. For $1 \leq i \leq n-1$, put $V_i = V_{t_i}$ and put $V_0 = U$.

Let $\gamma : [0, 1] \rightarrow X$ be any path such that $\gamma([t_i, t_{i+1}]) \subseteq V_i$ for $0 \leq i \leq n-1$ and $\gamma(t_i) = \alpha(t_i)$ for $0 \leq i \leq n$. We will now show that $[\gamma * \beta^-] \in H$. For simplicity, put $\delta_i = \alpha|_{[t_{i-1}, t_i]}$ and $\psi_i = \gamma|_{[t_{i-1}, t_i]}$. Note that $\alpha_{t_i} = \delta_1 * \delta_2 * \dots * \delta_i$ and that $\gamma_{t_i} = \psi_1 * \psi_2 * \dots * \psi_i$. Now express $[\gamma * \alpha^-] = [\psi_1 * \delta_1^-][\delta_1 * (\psi_2 * \delta_2^-) * \delta_1^-][\delta_1 * \delta_2 * (\psi_3 * \delta_3^-) * \delta_2^- * \delta_1^-] \dots [\delta_1 * \delta_2 * \dots * \delta_{n-1} * (\psi_n * \delta_n^-) * \delta_{n-1}^- * \dots * \delta_2^- * \delta_1^-]$. Define $\lambda_0 = \psi_1 * \delta_1^-$ and note this is a loop in U with $\lambda_0(0) = x_0$.

Since $\langle \delta_1 * (\psi_2 * \delta_2^-) \rangle \in \langle \delta_1, V_1 \rangle \cap p_H^{-1}(\alpha(t_1))$, we have $\langle \delta_1 * (\psi_2 * \delta_2^-) * \delta_1^- \rangle = \phi_{\delta_1^-}(\langle \delta_1 * (\psi_2 * \delta_2^-) \rangle) \in \langle c, U \rangle \cap p_H^{-1}(x_0)$. So $\langle \delta_1 * (\psi_2 * \delta_2^-) * \delta_1^- \rangle = \langle \lambda_1 \rangle$ for some loop λ_1 in U with $\lambda_1(0) = x_0$. Hence $[\delta_1 * (\psi_2 * \delta_2^-) * \delta_1^-] = h_1[\lambda_1]$ for some $h_1 \in H$.

Since $\langle \delta_1 * \delta_2 * (\psi_3 * \delta_3^-) \rangle \in \langle \delta_1 * \delta_2, V_2 \rangle \cap p_H^{-1}(\alpha(t_2))$, we have $\langle \delta_1 * \delta_2 * (\psi_3 * \delta_3^-) * \delta_2^- * \delta_1^- \rangle = \phi_{\delta_2^- * \delta_1^-}(\langle \delta_1 * \delta_2 * (\psi_3 * \delta_3^-) \rangle) \in \langle c, U \rangle \cap p_H^{-1}(x_0)$. Consequently, $[\delta_1 * \delta_2 * (\psi_3 * \delta_3^-) * \delta_2^- * \delta_1^-] = h_2[\lambda_2]$ for some loop λ_2 in U with $\lambda_2(0) = x_0$ and some $h_2 \in H$.

Inductively, we find that $[\gamma * \alpha^-] = [\lambda_0]h_1[\lambda_1]h_2[\lambda_2] \dots h_{n-1}[\lambda_{n-1}]$, where $h_i \in H$ for $1 \leq i \leq n-1$ and λ_i is a loop in U with $\lambda_i(0) = x_0$ for $0 \leq i \leq n-1$. Since each $[\lambda_i] \in \pi(c, U)$, we have $[\gamma * \alpha^-] = [\lambda_0]h_1[\lambda_1]h_2[\lambda_2] \dots h_{n-1}[\lambda_{n-1}] = \tilde{h}_1\tilde{h}_2 \dots \tilde{h}_{n-1}[\tilde{\lambda}_0][\tilde{\lambda}_1] \dots [\tilde{\lambda}_{n-2}][\lambda_{n-1}]$ for some $\tilde{h}_i \in H$ and loops $\tilde{\lambda}_i$ in U with $\tilde{\lambda}_i(0) = x_0$. Put $h = \tilde{h}_1\tilde{h}_2 \dots \tilde{h}_{n-1}$ and $\lambda = \tilde{\lambda}_0 * \tilde{\lambda}_1 * \dots * \tilde{\lambda}_{n-2} * \lambda_{n-1}$. Then $h \in H$ and λ is a loop in U with $\lambda(0) = x_0$ so that $[\gamma * \alpha^-] = h[\lambda]$. Hence, $\langle \gamma * \alpha^- \rangle = \langle \lambda \rangle \in \langle c, U \rangle \cap p_H^{-1}(x_0)$, so that $\langle \gamma * \alpha^- \rangle \neq \langle \beta * \alpha^- \rangle$. Consequently, $[\gamma * \beta^-] \notin H$. \square

Theorem 4.9 (Extension 3). *Let X be connected and locally path-connected, let $x_0 \in X$, H a locally quasinormal subgroup of $\pi_1(X, x_0)$, such that X is homotopically Hausdorff with respect to H and every monodromy ϕ_β is continuous for all paths β in X . Then the endpoint projection $p_H : \tilde{X}_H \rightarrow X$ is a fibration with unique path lifting.*

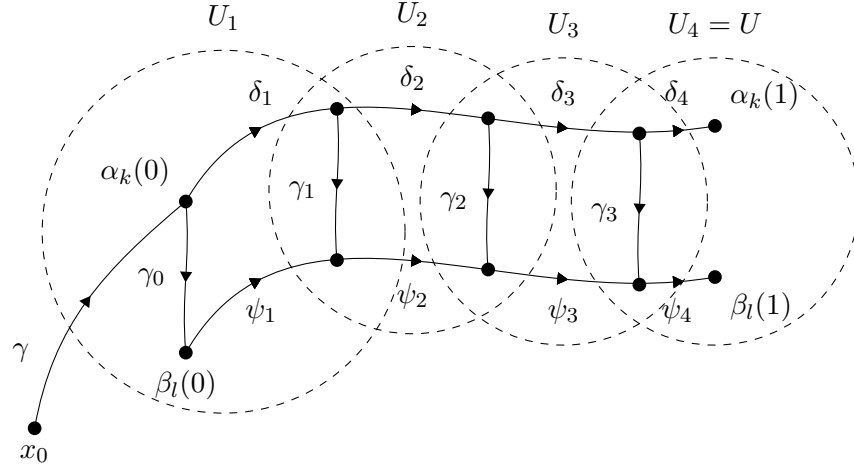
Proof. By Theorem 4.2, we only need to show that $p_H : \tilde{X}_H \rightarrow X$ is a fibration. Let $\tilde{f} : Y \rightarrow \tilde{X}_H$ and $F : Y \times [0, 1] \rightarrow X$ be continuous maps such that $p_H \circ \tilde{f}(y) = F(y, 0)$ for every $y \in Y$. We define a map $\tilde{F} : Y \times [0, 1] \rightarrow \tilde{X}$ as follows: for each $(y, k) \in Y \times [0, 1]$, $F|_{\{y\} \times [0, 1]} : \{y\} \times [0, 1] \rightarrow X$ defines a path $\alpha : [0, 1] \rightarrow X$ by $\alpha(s) = F(y, s)$. Define $\alpha_k(s) = \alpha(ks)$, so that $\alpha_k : [0, 1] \rightarrow X$ is the path along α from $F(y, 0)$ to $F(y, k)$. Put $\langle \beta \rangle = \tilde{f}(y)$. Now define $\tilde{F}(y, k) = \langle \beta * \alpha_k \rangle$. Then $\tilde{F}(y, 0) = \langle \beta \rangle = \tilde{f}(y)$ and $p_H \circ \tilde{F}(y, k) = \alpha_k(1) = \alpha(k) = F(y, k)$.

We need to show that \tilde{F} is continuous. Let $(y_0, k) \in Y \times [0, 1]$ and let α be the path such that $\alpha(t) = F(y_0, t)$, and define $\alpha_t(s) = \alpha(st)$. Put $\langle \gamma \rangle = \tilde{f}(y_0)$. Let U be an open neighborhood of $\alpha(k)$. Then $\langle \gamma * \alpha_k, U \rangle$ is a basic open neighborhood of $\tilde{F}(y_0, k)$. We will find open sets $M \subseteq Y$, $I \subseteq [0, 1]$ with $(y_0, k) \in M \times I \subseteq \tilde{F}^{-1}(\langle \gamma * \alpha_k, U \rangle)$.

Since H is locally quasinormal, we may assume that $H\pi(\gamma * \alpha_k, U) = \pi(\gamma * \alpha_k, U)H$. By continuity of monodromies, for each $t \in [0, k]$, there is an open subset U_t of X with $\alpha(t) \in U_t$ such that $\phi_\delta(\langle \gamma * \alpha_t, U_t \rangle \cap p_H^{-1}(\alpha(t))) \subseteq \langle \gamma * \alpha_k, U \rangle \cap p_H^{-1}(\alpha(k))$ where $\delta : [0, 1] \rightarrow X$ is the path along α from $\alpha(t)$ to $\alpha(k)$ given by $\delta(s) = \alpha(t + s(k - t))$. Using compactness of $[0, k]$ and Lemma 2.10, we find a finite subdivision $0 = t_0 < t_1 < t_2 < \dots < t_n = k$ such that, for every $1 \leq i \leq n - 1$, $\alpha([t_{i-1}, t_i]) \subseteq U_{t_i}$ and $\alpha([t_{n-1}, t_n]) \subseteq U$. For $1 \leq i \leq n - 1$ put $U_i = U_{t_i}$ and put $U_n = U$. For each $i \in \{1, 2, \dots, n - 1\}$, choose a path-connected open set V_i such that $\alpha(t_i) \in V_i \subseteq U_i \cap U_{i+1}$.

As in the Tube Lemma [12, Lemma 26.8], for every $1 \leq i \leq n$, there is an open subset N_i of Y with $y_0 \in N_i$ and an interval I_i , open in $[0, 1]$, with $[t_{i-1}, t_i] \subseteq I_i$ such that $F(N_i \times I_i) \subseteq U_i$. For every $1 \leq i \leq n - 1$, there is an open subset M_i of Y with $y_0 \in M_i$ and an interval J_i , open in $[0, 1]$, with $t_i \in J_i$ such that $F(M_i \times J_i) \subseteq V_i$. Since $\tilde{f}(y_0) = \langle \gamma \rangle$ and $\gamma(1) \in U_1$ and \tilde{f} is continuous, there is an open subset M_0 of Y with $y_0 \in M_0$ and $\tilde{f}(M_0) \subseteq \langle \gamma, U_1 \rangle$. Put $M = (\cap_{i=1}^n N_i) \cap (\cap_{i=0}^{n-1} M_i)$ and choose I to be an interval open in $[0, 1]$ with $t_n \in I \subseteq (t_{n-1}, 1] \cap I_n$.

Let $(y, l) \in M \times I$, and let $\beta : [0, 1] \rightarrow X$ be the path defined by $\beta(t) = F(y, t)$. Denote $\beta_t(s) = \beta(st)$ and, for simplicity, put $\psi_i = \beta|_{[t_{i-1}, t_i]}$ for $1 \leq i \leq n - 1$ and put $\psi_n = \beta|_{[t_{n-1}, l]}$. Then ψ_i lies in U_i and $\beta(t_i) \in V_i$. Since $\tilde{f}(y) \in \langle \gamma, U_1 \rangle$, $\tilde{f}(y) = \langle \gamma * \gamma_0 \rangle$ for some path γ_0 in U_1 with $\gamma_0(0) = \gamma(1) = \alpha(0)$. For each $1 \leq i \leq n - 1$, let γ_i be a path in V_i such that $\gamma_i(0) = \alpha(t_i)$ and $\gamma_i(1) = \beta(t_i)$. For $1 \leq i \leq n$, put $\delta_i = \alpha|_{[t_{i-1}, t_i]}$. Refer to the figure below, where the construction is shown for the case where $n = 4$.



Since $\tilde{f}(y) = \langle \gamma * \gamma_0 \rangle$, we have $\tilde{F}(y, l) = \langle \gamma * \gamma_0 * \psi_1 * \psi_2 * \cdots * \psi_n \rangle$. We wish to show that $\langle \gamma * \gamma_0 * \psi_1 * \psi_2 * \cdots * \psi_n \rangle \in \langle \gamma * \delta_1 * \delta_2 * \cdots * \delta_n, U \rangle$. That is, we wish to find a path ϵ in U and an element $h \in H$ such that $[\gamma * \gamma_0 * \beta_l] = h[\gamma * \alpha_k * \epsilon]$.

Express $[\gamma * \gamma_0 * \beta_l] = [\gamma * \gamma_0 * \psi_1 * \gamma_1^- * \delta_2 * \delta_3 * \cdots * \delta_n][\alpha_k^- * \gamma^-][\gamma * \delta_1 * \gamma_1 * \psi_2 * \psi_3 * \cdots * \psi_n]$. Since $\langle \gamma * \delta_1 * \delta_1^- * \gamma_0 * \psi_1 * \gamma_1^- \rangle \in \langle \gamma * \alpha_{t_1}, U_1 \rangle \cap p_H^{-1}(\alpha(t_1))$, we have that $\langle \gamma * \delta_1 * \delta_1^- * \gamma_0 * \psi_1 * \gamma_1^- * \delta_2 * \delta_3 * \cdots * \delta_n \rangle = \phi_{\delta_2 * \delta_3 * \cdots * \delta_n}(\langle \gamma * \delta_1 * \delta_1^- * \gamma_0 * \psi_1 * \gamma_1^- \rangle) = \langle \gamma * \alpha_k * \epsilon_1 \rangle$ for some loop ϵ_1 in U with $\epsilon_1(0) = \alpha(k)$. Hence $[\gamma * \gamma_0 * \psi_1 * \gamma_1^- * \delta_2 * \delta_3 * \cdots * \delta_n] = h_1[\gamma * \alpha_k * \epsilon_1]$ for some $h_1 \in H$. Put $a_1 = [\gamma * \alpha_k * \epsilon_1][\alpha_k^- * \gamma^-] \in \pi(\gamma * \alpha_k, U)$. Then $[\gamma * \gamma_0 * \beta_l] = h_1 a_1 [\gamma * \delta_1 * \gamma_1 * \psi_2 * \psi_3 * \cdots * \psi_n]$.

Since $\langle \gamma * \delta_1 * \delta_2 * \delta_2^- * \gamma_1 * \psi_2 * \gamma_2^- \rangle \in \langle \gamma * \alpha_{t_2}, U_2 \rangle \cap p_H^{-1}(\alpha(t_2))$, we have $\langle \gamma * \delta_1 * \delta_2 * \delta_2^- * \gamma_1 * \psi_2 * \gamma_2^- * \delta_3 * \delta_4 * \cdots * \delta_n \rangle = \phi_{\delta_3 * \delta_4 * \cdots * \delta_n}(\langle \gamma * \delta_1 * \delta_2 * \delta_2^- * \gamma_1 * \psi_2 * \gamma_2^- \rangle) = \langle \gamma * \alpha_k * \epsilon_2 \rangle$ for some loop ϵ_2 in U with $\epsilon_2(0) = \alpha(k)$. Hence $[\gamma * \delta_1 * \gamma_1 * \psi_2 * \gamma_2^- * \delta_3 * \delta_4 * \cdots * \delta_n] = h_2[\gamma * \alpha_k * \epsilon_2]$ for some $h_2 \in H$. We then put $a_2 = [\gamma * \alpha_k * \epsilon_2][\alpha_k^- * \gamma^-] \in \pi(\gamma * \alpha_k, U)$, and thus obtain $[\gamma * \gamma_0 * \beta_l] = h_1 a_1 h_2 a_2 [\gamma * \delta_1 * \delta_2 * \gamma_2 * \psi_3 * \psi_4 * \cdots * \psi_n]$.

Inductively, we find $[\gamma * \gamma_0 * \beta_l] = h_1 a_1 h_2 a_2 \cdots h_{n-1} a_{n-1} [\gamma * \delta_1 * \delta_2 * \cdots * \delta_{n-1} * \gamma_{n-1} * \psi_n]$. Using local quasinormality, $a_{i-1} h_i = \tilde{h}_i \tilde{a}_{i-1}$ with $\tilde{h}_i \in H$ and $\tilde{a}_{i-1} = [\gamma * \alpha_k * \tilde{\epsilon}_{i-1} * \alpha_k^- * \gamma^-] \in \pi(\gamma * \alpha_k, U)$ for some loop $\tilde{\epsilon}_{i-1}$ in U . We put $\epsilon_n = \delta_n^- * \gamma_{n-1} * \psi_n$, and note ϵ_n is a path in U . Put $\epsilon = \tilde{\epsilon}_1 * \tilde{\epsilon}_2 * \cdots * \tilde{\epsilon}_{n-2} * \epsilon_{n-1} * \epsilon_n$ and $h = h_1 \tilde{h}_2 \cdots \tilde{h}_{n-1}$. Then ϵ is a path in U and $h \in H$ such that $[\gamma * \gamma_0 * \beta_l] = h[\gamma * \alpha_k * \epsilon]$. \square

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